Review Study on Solving the Singular Linear Algebraic Systems Using the Drazin Inverse of the Matrix

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الملخص

معكوس Drazin لمصفوفة معطاة يعني مصفوفة موجودة لمصفوفة شاذة ومربعة والتي لها بعض خواص المعكوس الضربي للمصفوفات والتي تتطابق معها في حاله المصفوفة غير الشاذة، وهذا يعني أن فهرس المصفوفة المعطاة تساوي صفراً. إن دراسة الأنظمة الجبرية الخطية هي إحدى مجالات الدراسة المهمة في الجبر الخطي. نحن نتأمل الأنظمة الجبرية الخطية الشاذة على الصورة $Ax = b$ حيث A مصفوفة مربعة شاذة بمداخل من حقل الأعداد المركبة $\mathbb{C}$، x و b متجهين بمداخل من حقل الأعداد المركبة أيضاً. نحن ندرس حل هذه الأنظمة باستخدام معكوس Drazin للمصفوفة، لذلك نحن نعطي الحل العام وأصغر حل تربيعي معمم والحد الأدنى الوحيد لأصغر حل تربيعي للنظام المعطى. النتائج مأخوذة من المراجع المشار إليها في نهاية البحث، نحن نعطي بعض الأمثلة لتوضيح دراستنا.

Abstract

By the Drazin inverse of given matrix, we mean a matrix that exists for a square singular matrix that has some of the properties of the multiplicative inverse of matrices and that agrees with it when given matrix is non-singular, that means the index of a given matrix equals zero. Study the linear algebraic systems is one of the important study fields of linear algebra. We consider the singular linear algebraic systems on the form $Ax = b$, where $A \in \mathbb{C}^{n\times n}$ is singular, and x, b are vectors in $\mathbb{C}^n$. We study solving these systems using the Drazin inverse of the matrix. For that, we give the general solution, the

generalized least squares solution and the unique minimal least squares solution for a given system. The results are taken from the mentioned references. We give some examples to illustrate our study.
Keywords: Singular linear system, the Drazin inverse, minimal least squares solution.

1. Introduction

The singular linear algebraic system started to attract attention of researches at the end of1970s. Many results of classical linear system generalized to the singular linear system.
The singular linear system of equations

$$Ax = b; \quad A \in \mathbb{C}^{n\times n}, \qquad x, b \in \mathbb{C}^n \tag{1}$$

has been exploited in several contexts. It is consistent (i.e., has solution for x) if and only if b is in the range of A. If b is not in the range of A then $b - Ax$ is nonzero for all $x \in \mathbb{C}^n$ [1,2].

The purpose of this paper is to study the singular linear algebraic system (1) when b belonges to the range of A^k, and when b does not belongs to the range of A^k. For that study, we use the Drazin inverse of a singular square matrix to find the solutions of the system (1). We give the general solution and the unique minimal least squares solation for (1) [3-6]. We give examples to illustrate our study.

Throughout this paper, $\mathbb{C}^n$ is an n-dimension complex vector space, $\mathbb{C}^{n\times n}$ denotes the complex space of $n \times n$-matrices (square matrices), A^{-1}denotes the multiplicative inverse of the matrix A, $\Re(A)$ denotes the range of A and $\aleph(A)$ denotes the null space of A and Tr(A) denotes the trace of A which is the summation of diagonal elements of A.

2. Preliminaries

In this section, we give important concepts about the Drazin inverse of a square matrix. For more details, see [1,7].

Definition 1. Let $A \in \mathbb{C}^{m\times n}$, and B is in the reduced graded row form and row equivalent to A, then the number of non-zero rows of B is the rank of A, and denoted by rank (A).

Definition 2. Let $A \in \mathbb{C}^{n\times n}$, then the smallest non–negative integer$k$ such that *rank* (A^k) = *rank* (A^{k+1}) is called the index of

A and denoted by $Ind(A)$. Note that $Ind(0) = 1$, where 0 is the zero matrix, and if $Ind(A) = 0$ then A is invertible, that is, $A^D = A^{-1}$.

Definition 3. Let $A \in \mathbb{C}^{n\times n}$ with $Ind(A) = k$, then the Drazin inverse of A is defined to be the unique matrix A^D such that

i. $A^D A\, A^D = A^D$.
ii. $A^D A = A\, A^D$, and
iii. $A^{k+1} A^D = A^k$.

Definition 4. [1] A matrix H $\in \mathbb{C}^{n\times n}$ is said to be in hermite echelon form if its elements h_{ij} satisfy the following conditions.

i. H is upper triangular (i.e. h_{ij}= 0 where i >j) .
ii. h_{ii} is either 0 or 1.
iii. If h_{ii}= 0, then $h_{il} = 0$ for every l, 1≤ $l \le n$.
iv. If h_{ii}= 1, then h_{li}= 0 for $l \neq i$.

Definition 5. The least squares solution with the minimum norm of the system (1) is the vector $x^0 \in \mathbb{C}^n$ satisfying

$$\|x^0\| = min_{u\in\mathbb{C}^n}\{\|u\|: \|Au - b\| = min_{x\in\mathbb{C}^n}\|Ax - b\|\}.$$

Definition 6. The W–norm is defined as

$$\|x\|_W = \|W^{-1}x\|_2\, , \forall x \in \mathbb{C}^n$$

where W is a non-singular matrix that transforms A into its Jordan canonical form, and $\|.\|_2$ denotes the Euclidean norm. Thus, the minimal W-norm least squares solution is

$$\|W^{-1}x\| = min_{u\in\mathbb{C}^n}\{\|u\|: \|Au - b\| = min_{x\in\mathbb{C}^n}\|Ax - b\|\}.$$

Algorithm 1. [1,7] (Computation of A^D where $A \in \mathbb{C}^{n\times n}$ and $Ind(A) = k$)

i. Let p be an integer such that $p \ge k$. (p can always be taken to equal to n if no smaller value can be determined). If $A^p = 0$, then $A^D = 0$. Thus we assume $A^p \neq 0$.
ii. Row reduce A^p to its Hermite echelon form H_{A^p}. The sequence of reducing matrices need not be saved.
iii. By noting the position of the non-zero diagonal elements in H_{A^p}, select the distinguished columns from A^p and call them $v_1, v_2, \ldots, v_r$.
iv. Form the matrix $I - H_{A^p}$ and save its non-zero columns. Call them $v_{r+1}, v_{r+2}, \ldots, v_n$.

v. construct the non-singular matrix $P = [v_1|\dots|v_r|v_{r+1}|\dots|v_n]$.
vi. Compute P^{-1}.
vii. From the product $P^{-1}AP$. This matrix will be in the form $P^{-1}AP = \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix}$ where C is non-singular and N is nilpotent.
viii. Compute C^{-1}.
ix. Compute A^D by forming the product $A^D = P\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}$.

The Drazin inverse of a matrix can be represented by the Jordan canonical form as in the following theorem.

Theorem 1. [1] If $A \in \mathbb{C}^{n\times n}$ is such that $Ind(A)= k > 0$, then there exists a nonsingular matrix P such that $A = P\begin{pmatrix} C & 0 \\ 0 & N \end{pmatrix}P^{-1}$, where C is non-singular, and N is nilpotent of index k. furthermore, if P, C and N are any matrices satisfying the above conditions, then

$$A^D = P\begin{pmatrix} C^{-1} & 0 \\ 0 & 0 \end{pmatrix}P^{-1}.$$

3. Solutions of Singular Linear Algebraic Systems

There are many methods to solve the singular linear algebraic system (1). Among these methods are using Grammar's rule [4], Krylov subspace methods [3]. Recently, we collect some results that suggested in some references [3-6], when index of A is greater than one. We give the solutions of the system (1) for some cases.

3.1 The General Solution

In this subsection, we discuss using the Drazin inverse to solve the singular linear system on the form (1) in the case $b \in \mathfrak{R}(A^k)$ and index of A is arbitrary. In the case $b \in \mathfrak{R}(A^k)$ and index of A is arbitrary then the general solution of the system (1) is

$$x = A^D b + Wz\,; \quad W = I - AA^D \qquad (2)$$

where $z \in \mathfrak{R}(A^{k-1}) + \aleph(A)$, W is called the projection, AA^D is called the Drazin inverse solution, If $z \notin \mathfrak{R}(A^{k-1}) + \aleph(A)$, then the vector $A^D b + Wz$ is not a solution of the system (1). We give example to illustrate the solution when $b \in \mathfrak{R}(A^k)$. Note that

$$\Re(A^k) \oplus \aleph(A^k) = \mathbb{C}^n.$$

if and only if $Ind(A)= k$.

Example 1. Let the following singular system

$$2x_1 + 4x_2 + 6x_3 + 5x_4 = 8$$
$$x_1 + 4x_2 + 5x_3 + 4x_4 = 7$$
$$-x_2 - 6x_3 = -3$$
$$-x_1 - 2x_2 - 3x_3 - 3x_4 = -3.$$

Note that

$$A = \begin{pmatrix} 2 & 4 & 5 & 6 \\ 1 & 4 & 5 & 4 \\ 0 & -1 & -1 & 0 \\ -1 & -2 & -3 & -3 \end{pmatrix}, Ind(A) = 2, b = \begin{pmatrix} 8 \\ 7 \\ -3 \\ -3 \end{pmatrix} \in \Re(A^2)$$

and

$$A^D = \begin{pmatrix} 3 & -1 & 2 & 2 \\ 2 & 1 & 3 & 3 \\ -1 & 0 & -1 & -1 \\ -1 & 0 & -1 & -1 \end{pmatrix}, A^D b = \begin{pmatrix} 5 \\ -1 \\ -2 \\ -2 \end{pmatrix}, I - AA^D$$
$$= \begin{pmatrix} -2 & -2 & -6 & -6 \\ -2 & -2 & -5 & -5 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}.$$

Hence, the general solution of the given system is given by

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 5 \\ -1 \\ -2 \\ -2 \end{pmatrix} + \begin{pmatrix} -2 & -2 & -6 & -6 \\ -2 & -2 & -5 & -5 \\ 1 & 1 & 3 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix} z,$$

$z \in \Re\,(A^{k-1}) + \aleph(A)$.

3.2 Generalized Least Squares Solution

In the system (1), if index of A is arbitrary, and $b \notin \Re(\,A^k)$ then

$$x = A^D b + Wz;\;\; W = I - AA^D \qquad (3)$$

Where $z \in \Re(A^{k-1}) + \aleph(A)$ is a generalized least squares solution in $z \in \Re\,(A^k) + \aleph(A)$ to the system (1) with respect to $\|x\|_W$. But $x = A^D b + (I - AA^D)z$ is not a solution of the system (1) even if $b \in \Re(A) \setminus \Re(A^k)$. Note that, the vector (3) is a generalized least squares solution in $\Re(A^k) + \aleph(A)$ of the system (1) with respect to the norm$\|x\|_W$.

3.3 The Minimal W-norm Solution

In subsection (3.2), we showed that

$$x = A^D b + Wz; \;\; W = I - AA^D$$

is the generalized least squares solution of the system (1) in the case $b \notin \Re(A^k)$, where $A^D b$ is the Drazin inverse solution of the system (1). But when

$$\|A^D b\|_W \leq \|x\|_W$$

where x is given above, then $A^D b$ is called the minimal W-norm least squares solution of the system(1). Note that, $A^D\, b$is not a solution of system (1), but it is the unique minimal W-norm least squares solution [4]. In the fact $A^D b$ is a solution of system (1) if and only if $b \in \Re(A^k)$ and it is the unique solution in $\Re(A^k)$ as showed Campbell [1].

Conclusion

Our study showed the importance of using the Drazin inverse of the matrices to solve the singular linear algebraic system when $b \in \Re(A^k)$ and $b \notin \Re(A^k)$. In other words, we study that system to get the general solution, the generalized least squares solution and the minimal W-norm least squares solution.

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